Cayley graphs: Symmetries, quantizations, duality

Quantum group seminar

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Cayley graphs

DEF — Cayley graph

Group Γ, set $S \subset \Gamma$ $\quad\rightarrow\quad$ $V = \Gamma$, $E = \{(g, kg) \mid g \in \Gamma, k \in S\}$.

$$A_{gh} = \begin{cases} 1 & g = kh \text{ for some } g \in \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

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EX

Hypercube

Take $\Gamma = \mathbb{Z}_2^n = \{(i_1, \ldots, i_n) \mid i_1, \ldots, i_n = 0, 1\}$, denote $\epsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ and take $S = \{\epsilon_i\}_{i=1}^n$.

Quantum symmetries of Cayley graphs

DEF

Quantum automorphism group of a graph

$$C(\mathrm{Aut}^+ G) = C^*(u_{ij} \mid uu^* = 1 = u^*u, m(u \otimes u) = um, u\eta = \eta, uA = Au).$$

[Banica '05]

Cayley graphs

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PROP

If Γ is abelian, then $\operatorname{Irr}\Gamma$ forms an eigenbasis for A. Given $\chi \in \operatorname{Irr}\Gamma$, we have

$$\lambda_\chi = \sum_{k \in S} \chi(k^{-1}).$$

[Lovász '75, Babai '79]

Cayley graphs

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EXAMPLE

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$$\chi_{(j_1,\ldots,j_n)}(i_1, \ldots, i_n) = (-1)^{i_1 j_1 + \cdots + i_n j_n}$$

$$\lambda_{(j_1,\ldots,j_n)} = \sum_{k=1}^{n} (-1)^{j_k} = n - 2\#\{\text{non-zero } j_i\text{'s}\}.$$

$\longrightarrow$ Eigenspaces $V_k = \operatorname{span}\{\chi_{(j_1,\ldots,j_n)} \mid \text{exactly } k\ j_i\text{'s are non-zero}\}$.

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Intertwiners in the basis (χ_g):

$$A\chi_g = \lambda_g \chi_g, \qquad m(\chi_g \otimes \chi_h) = \chi_{gh}, \qquad \eta = \chi_e.$$

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THM

$$\mathrm{Aut}^+ Q_n = O_n^{-1}$$

[Banica–Bichon–Collins '07]

Some other Cayley graphs

Denote $\mathbb{Z}_m^n = \{(i_1, \ldots, i_n) \mid i_1, \ldots, i_n = 0, \ldots, m-1\}$, $\epsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, $\iota = (1, 1, \ldots, 1)$

Hypercube Q_n

$\Gamma = \mathbb{Z}_2^n$, $S = \{\epsilon_i\}_{i=1}^n$ $\rightarrow$ inv. subspaces $V_k = \text{span}\{\chi_{\epsilon_{i_1} \cdots \epsilon_{i_k}}\}_{i_1 < \cdots < i_k}$ (i.e. $V_1 = \text{span}\{\chi_{\epsilon_i}\}_{i=1}^n$)

Halved/squared hypercube $\frac{1}{2}Q_{n+1} = Q_n^2$

$\Gamma = \mathbb{Z}_2^n$, $S = \{\epsilon_i\}_{i=1}^n \cup \{\epsilon_i\epsilon_j\}_{i<j}$ $\rightarrow$ inv. subspaces $V_k + V_{n+1-k}$ (i.e. $\tilde{V}_1 = \text{span}\{\chi_{\epsilon_i}\}_{i=1}^n \cup \{\chi_\iota\}$)

THM

$$\text{Aut}^+ \frac{1}{2}Q_{n+1} = SO_{n+1}^{-1}$$

[G. '22]

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Folded hypercube FQ_{n+1}

$\Gamma = \mathbb{Z}_2^n$, $S = \{\epsilon_i\}_{i=1}^n \cup \{\iota\}$ $\rightarrow$ inv. subspaces $V_{2k-1} + V_{2k}$ (i.e. $\tilde{V}_1 = \mathrm{span}\{\chi_{\epsilon_i}\}_{i=1}^n \cup \{\chi_{\epsilon_i\epsilon_j}\}_{i<j}$)

THM

$$\mathrm{Aut}^+ FQ_{n+1} = PO_{n+1}^{-1}$$

[Schmidt '20, G. '22]

Some other Cayley graphs

Denote $\mathbb{Z}_m^n = \{(i_1,\ldots,i_n) \mid i_1,\ldots,i_n = 0,\ldots,m-1\}$, $\epsilon_i = (0,\ldots,0,1,0,\ldots,0)$, $\iota = (1,1,\ldots,1)$

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Hamming graph $H(n,m)$

$\Gamma = \mathbb{Z}_m^n$, $S = \{\epsilon_i^j\}_{i=1,\ldots,n;j=1,\ldots,m}$ $\rightarrow$ inv. subspaces $V_k = \mathrm{span}\{\chi_{\epsilon_{i_1}^{j_1}\cdots\epsilon_{i_k}^{j_k}}\}_{i_1<\cdots<i_k;j_1,\ldots,j_k=1,\ldots,m-1}$ (i.e. $V_1 = \mathrm{span}\{\chi_{\epsilon_i}^j\}_{i=1,\ldots n;j=1,\ldots,m-1}$)

THM

$$\mathrm{Aut}^+ H(n,n) = S_m^+ \wr S_n$$

[G. '22]

Anticommutative hypercube

For the classical hypercube, we have vertex set $\Gamma = \mathbb{Z}_2^n$, so

$$\begin{aligned} C(\Gamma) &= C^*(e_i \mid e_i e_j = \delta_{ij} e_i, e_i^* = e_i) \\ &= C^*(\chi_g, g \in \Gamma \mid \chi_g \chi_h = \chi_{gh}, \chi_{g^{-1}} = \chi_g^*) \\ &= C^*(\chi_i \mid \chi_i \chi_j = \chi_j \chi_i, \chi_i^2 = 1, \chi_i^* = \chi_i) \end{aligned}$$

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Twist:

$$C(\check{\Gamma}) = C^*(\chi_i \mid \chi_i \chi_j = -\chi_j \chi_i, \chi_i^2 = 1, \chi_i^* = \chi_i)$$

Twisting Cayley graphs of abelian groups

DEF

A *finite quantum space* X is given by a finite-dimensional C*-algebra $C(X)$. Every such algebra is equipped by a unique structure of a special symmetric Frobenius algebra. The associated Hilbert space is denoted by $l^2(X)$. It induces an operation on linear operators $A, B\colon l^2(X) \to l^2(X)$ by $A \bullet B = m(A \otimes B)m^\dagger$ called the *Schur product*.

DEF

A *directed quantum graph* is a quantum space X equipped with an adjacency operator $A\colon l^2(X) \to l^2(X)$ satisfying $A \bullet A = A = A^*$.

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DEF

A *directed quantum graph* is a quantum space X equipped with an adjacency operator $A\colon l^2(X) \to l^2(X)$ satisfying $A \bullet A = A = A^*$.

Recall $\quad C(\Gamma) = C^*(\chi_g, g \in \Gamma \mid \chi_g \chi_h = \chi_{gh}, \chi_{g^{-1}} = \chi_g^*)$.

For any unitary bicharacter σ on Γ, define $\quad C(\check{\Gamma}) = C^*(\chi_g, g \in \Gamma \mid \sigma_{gh} \chi_g \chi_h = \chi_{gh}, \chi_{g^{-1}} = \chi_g^*)$.

Define $A\colon C(\check{\Gamma}) \to C(\check{\Gamma})$ by $A\chi_g = \lambda_g \chi_g$ with $\lambda_g = \sum_{k \in S} \chi_g(k^{-1})$.

THM

Let Γ be a finite abelian group, σ a unitary bicharacter on Γ and $S \subset \Gamma$. Then the adjacency operator given as above defines a directed quantum graph on $\check{\Gamma}$, which is quantum isomorphic to the classical Cayley graph $\mathrm{Cay}(\Gamma, S)$. Its quantum automorphism group is the 2-cocycle twist of the quantum automorphism group $\mathrm{Aut}^+\,\mathrm{Cay}(\Gamma, S)$ of the classical Cayley graph $\mathrm{Cay}(\Gamma, S)$.

[G. '22]

Association schemes

DEFINITION — Association scheme

Consider the set $X = \{1, \ldots, n\}$ with $n \in \mathbb{N}$. A d-class *association scheme* over X is a set $\mathbf{A} = \{A_0, A_1, \ldots, A_d\}$ of $n \times n$ matrices A_i such that

1. All A_i have entries in $\{0, 1\}$,
2. $A_0 = I$,
3. $\sum_{i=0}^{d} A_i = J$, where J is the all ones matrix
4. $A_i^{\mathrm{T}} \in \mathbf{A}$ for every i,
5. $A_i A_j \in \operatorname{span} \mathbf{A}$ for every i, j,

DEF — Distance regular graph

A graph is called *distance regular* if partitioning pairs of vertices according to their distance forms an association scheme.

DEF — Translation association scheme

A *translation association scheme* is an association scheme $\mathbf{A} = \{A_i\}$ over some group Γ such that all A_i are Cayley.

Translation association schemes and duality

DEF

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DEF

A *translation association scheme* is an association scheme $\mathbf{A} = \{A_i\}$ over some group Γ such that all A_i are Cayley (corresponding to some sets $S_i \subset \Gamma$).

The characters χ_g of Γ form a common eigenbasis of $\mathbf{A}$. So, we can denote the common eigenspaces by

$$V_j = \operatorname{span}\{\chi_g \mid g \in \tilde{S}_j\}.$$

Claim. The sets $\tilde{S}_j$ define a new translation association scheme called the *dual*.

Duality for groups vs. schemes

Abelian groups:

$l^2(\Gamma) = \mathrm{span}\{e_g\}_{g\in\Gamma}$ is equipped with two products:

$$e_g \cdot e_h = \delta_{gh} e_g, \qquad e_g * e_h = e_{gh}.$$

Change of basis (Fourier transform) swaps the products:

$$\chi_g \cdot \chi_h = \chi_{gh}, \qquad \chi_g * \chi_h = n\delta_{gh}\chi_g.$$

Group schemes:

DEF Let Γ be a group. We associate to it a *group scheme* given by $\mathbf{A} = \{A^g\}_{g\in\Gamma}$, where A^g is the Cayley graph of $S = \{g\}$.

Observe: We have $A^g e_h = e_{gh} = e_g * e_h$. Moreover,

$$A^g A^h = A^{gh}, \qquad A^g \bullet A^h = \delta_{gh} A^g.$$

Duality for groups vs. schemes

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Group schemes:

Taking $S = \{g\}$ means $A^g e_h = e_g * e_h$. Then

$$A^g A^h = A^{gh}, \qquad A^g \bullet A^h = \delta_{gh} A^g.$$

Translation schemes

In general, taking any S, we have $Ae_h = \sum_{k \in S} e_{kh} = \left(\sum_{k \in S} e_k\right) * e_h$.

Duality for quantum groups and quantum group schemes

DEFINITION

Cocommutative quantum association scheme

Let X be a finite quantum space. A d-class *cocommutative quantum association scheme* (CQAS) is a set $\mathbf{A} = \{A_0, A_1, \ldots, A_d\}$ with $A_i\colon l^2(X) \to l^2(X)$ such that

1. $A_i = A_i^*$, $A_i \bullet A_i = A_i$ for every i,
2. $A_0 = I$,
3. $\sum_{i=0}^{d} A_i = J$,
4. $A_i^\dagger \in \mathbf{A}$ for every i,
5. $A_i A_j \in \operatorname{span} \mathbf{A}$ for every i, j.

[G. '24]

DEF

Finite quantum group

A *finite quantum group* Γ is determined by a finite-dimensional Hopf $*$-algebra $C(\Gamma)$

→ two kinds of multiplication in $l^2(G)$: $x \bullet y, \quad x * y.$

We define the *dual quantum group* by swapping the two products.

Duality for quantum groups and quantum group schemes

Cocommutative quantum association scheme

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Cayley quantum graph

DEF

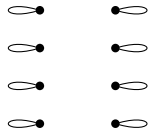
For any $p \in l^2(\Gamma)$ such that $p \bullet p = p = p^*$, the map $x \mapsto p * x$ defines a quantum graph. We call it the *Cayley quantum graph*.

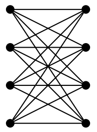
Quantum translation association scheme

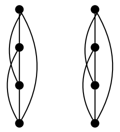
DEF

A *translation* QCAS over some quantum group Γ means that all A_i are Cayley.

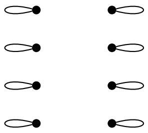
Example: Complete bipartite graph



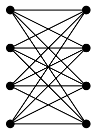




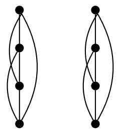
Example: Complete bipartite graph – Cayley w.r.t. $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_2$



$S_0 = \{(0,0)\}$



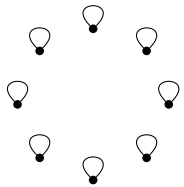
$S_1 = \{(i,1) \mid i \in \mathbb{Z}_n\}$



$S_2 = \{(i,0) \mid i \in \mathbb{Z}_n \setminus \{0\}\}$

$$\lambda_{(i,j)} = \sum_{(a,b)\in S} \tau_{(i,j)}(a,b)^{-1} = (-1)^j \sum_{a=0}^{n-1} \omega^{ia} = \begin{cases} 0 & i \neq 0, \\ n & i = 0, j = 0, \\ -n & i = 0, j = -1. \end{cases}$$

$\tilde{S}_0 = \{(0,0)\}$



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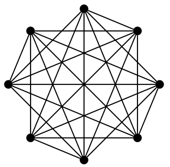

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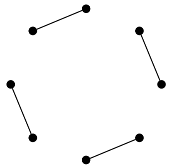

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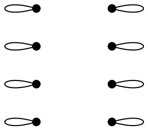
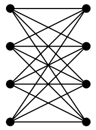


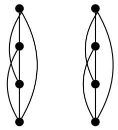
$S_0 = \{(0,0)\}$

$\alpha_0 = e_{(0,0)}$



$S_1 = \{(i,1) \mid i \in \mathbb{Z}_n\}$

$\alpha_1 = \sum_{i \in \mathbb{Z}_n} e_{(i,1)}$



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$\tilde{S}_0 = \{(0,0)\}$

$\epsilon_0 = \chi_{(0,0)} = \alpha_0 + \alpha_1 + \alpha_2$

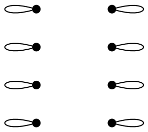
$\tilde{S}_1 = \{(i,j) \mid i \neq 0\}$

$\epsilon_1 = \sum_{i \neq 0} \chi_{(i,j)} = (2n-2)\alpha_0 - 2\alpha_2$

$\tilde{S}_2 = \{(0,-1)\}$

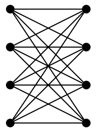
$\epsilon_2 = \chi_{(0,1)} = \alpha_0 - \alpha_1 + \alpha_2$

Example: Complete bipartite graph – Cayley w.r.t. D_n



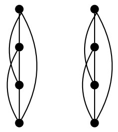
$S_0 = \{e\}$

$\alpha_0 = e_e$



$S_1 = \{sr^k \mid k = 0, \dots, n-1\}$

$\alpha_1 = \sum_{k=0}^{n-1} e_{sr^k}$



$S_2 = \{r^k \mid k = 1, \dots, n-1\}$

$\alpha_2 = \sum_{k=1}^{n-1} e_{r^k}$

Eigenvalues and eigenspaces do not depend on Γ, so the following is a basis of $\operatorname{span}\{\alpha_0, \alpha_1, \alpha_2\} \subset \mathbb{C}D_4$ by orthogonal projections:

$$\epsilon_0 = \alpha_0 + \alpha_1 + \alpha_2 = \eta \qquad \epsilon_1 = (2n-2)\alpha_0 - 2\alpha_2 \qquad \epsilon_2 = \alpha_0 - \alpha_1 + \alpha_2$$

Hence, we have the adjacency matrices acting by

$$g \mapsto g \ \forall g \in D_4$$

$$e \mapsto (2n-2)e \qquad r^i \mapsto -2r^i \qquad sr^i \mapsto 0$$

$$e \mapsto e \qquad r^i \mapsto r^i \qquad sr^i \mapsto -sr^i$$